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Summary: The cross-section of an arbitrarily-shaped waveguide is transformed into a rectangle. The rectangular guide is filled with a nonuniform anisotropic medium with such a distribution that the propagation properties are the same. Feenberg's perturbation method and the Rayleigh-Ritz method can be used for determining the propagation characteristics and the field distribution of the rectangular guide. The propagation constants of parabolic and elliptic guides are determined by both methods and compared with the exact values.

Author

Introduction

The common rectangular and circular waveguides often do not have the desired properties which can be obtained by other cross-sections. The ridged waveguide is an example for such a guide. The compution of the properties and characteristics of such nonconventional waveguides can be carried out by conformal mapping as described by Tischer and Yee in preceding reports.[1,2]

If the cross-section of an arbitrarily shaped air filled waveguide can be transformed by conformal mapping into a rectangle, the analysis of the arbitrarily shaped guide can be replaced by that of a rectangular guide filled with a nonuniform and anisotropic medium. The propagation properties of the latter guide can be computed by methods of solving partial differential equations of second order with variable coefficients. Feenberg's perturbation method was described previously.

In cases where the Feenberg pertubation method is slowly convergent or not convergent at all, the Rayleigh-Ritz method may be used for determining the characteristics of the guide.

In this report the parabolic and the elliptic guide are considered as examples for non-conventionally shaped waveguides. First, the parabolic guide with a vane is analyzed and its characteristics computed by both Feenberg's and Rayleigh-Ritz methods. The results are compared with the known exact solutions. Next, the propagation constants for a number of vane modes of a semi-elliptical guide are calculated.

Conformal Mapping

Two cross-sections of waveguides, one arbitrarily shaped and one rectangular, are assumed. The two cross-sections form lines of constant coordinates in two complex

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[1] F. J. Tischer, Proc. IEEE, Vol. 51, pp. 1050, July 1963.

[2] F. J. Tischer and H. Y. Yee, UARI Research Report No. 12, University of Alabama Research Institute (1964).

planes as shown in Fig. 1. Points along the cross-sections and boundaries are interrelated by a complex function

$$R = p + jq = f(Z),$$

where $R = p + jq$ is the plane with the air filled cross-section, and $Z = x + jy$ is the plane with non-uniformly filled cross-section.

The original air-filled waveguide is bounded by a perfect conductor. The rectangular equivalent guide which has perfectly conducting walls is filled by a non-uniform anisotropic medium as shown in reference 2. The properties of the medium are described by a tensor permittivity and permeability as follows:

$$\epsilon = \epsilon_o \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & h^2(x,y) \end{vmatrix} ; \qquad \mu = \mu_o \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & h^2(x,y) \end{vmatrix} .$$

The wave equation for the rectangular guide is given by

$$\triangle \Psi + k^2 h^2 (x,y) \Psi = 0 \qquad (1)$$

for time varying fields (e^{jwt}), where $\triangle$ is the two-dimensional Laplacian operator. The propagation constant k is given by

$$k^2 = k_o^2 - k_z^2 ,$$

$$k_o^2 = w^2 \mu_o \epsilon_o ,$$

$$k_z^2 = \left(\frac{2\pi}{\lambda g}\right)^2 .$$

The quantity k_z is the longitudinal propagation constant and λg is the guide wavelength. The scale factor h of the confomal mapping is given by

$$h(x,y) = \left| f'(z) \right| .$$

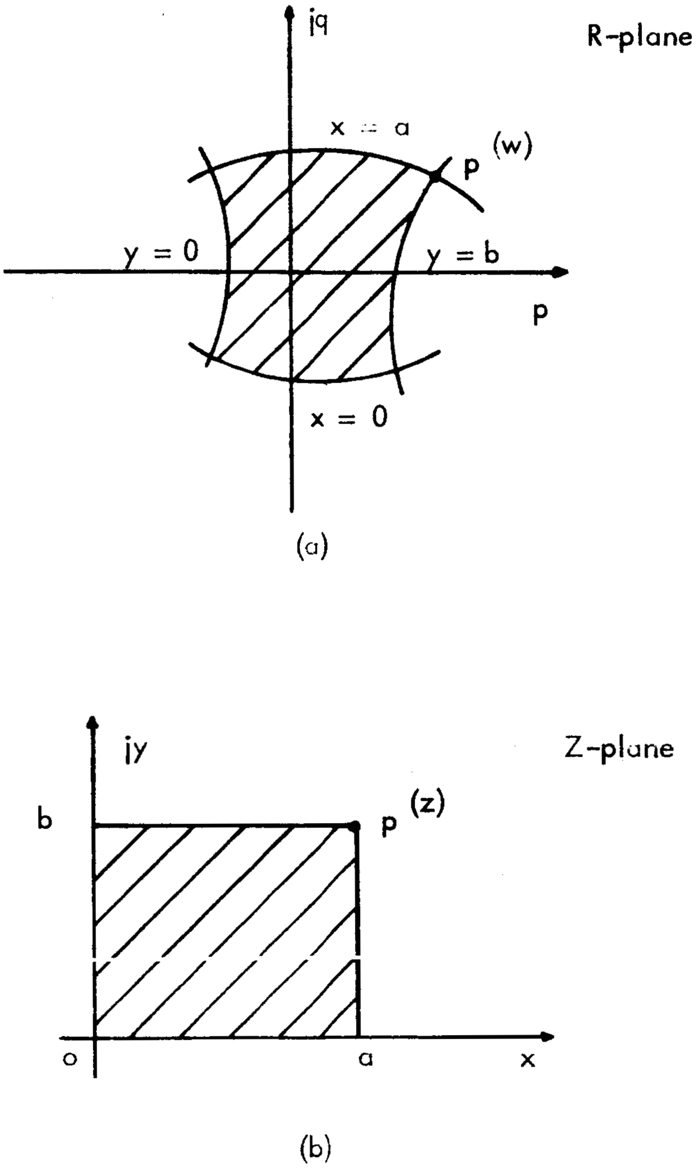

Fig. 1 - Cross-sections of waveguides in corresponding amplex planes. (a) Arbitrary shape in the R-plane. (b) The corresponding rectangular cross-section in the Z-plane.

The function ψ stands for the hypothetical components of the electrical and magnetic field intensities. For TE wave modes $\psi = H_z$, for TM wave modes $\psi = E_z$. The scalare function ψ is subject to the boundary conditions:

$$\psi(0,y) = \psi(a,y) = \psi(x,0) = \psi(x,b) = 0 \qquad (2a)$$

for TM modes. The corresponding conditions for TE modes are

$$\frac{\partial}{\partial x}\psi(o,y) = \frac{\partial}{\partial x}\psi(a,y) = \frac{\partial}{\partial y}\psi(x,o) = \frac{\partial}{\partial y}\psi(x,b) = 0 \ . \qquad (2b)$$

The next step consists in finding solutions of the wave equation [Eq. (1)] taking into account the boundary conditions [Eqs. (2)].

<u>The Feenberg Perturbation Method</u>

Except for four well known cases, Eq. (1) is nonseparable and no exact solutions for arbitrary boundaries are known. Therefore, approximate techniques must be used. Two methods of solving the wave equation by which the cut off frequencies, the propagation constants, and the fields distributions can be computed are described. They are Feenberg's perturbation method [2,3] and Rayleigh-Ritz method.

Feenberg's method requires an expansion of the scalar function ψ in terms of a complete set of orthonormal functions $\{\phi_q\}$, i.e. set

$$\psi = \sum_q A_q \phi_q \ , \qquad (3)$$

where ϕ_q satisfies the boundary conditions as ψ does,

$$\iint_s \phi_r \phi_q \, dS = \delta_{rq} \ ,$$

[3] P. M. Morse and H. Feshbach, <u>Methods of Theoretical Physics</u>, (McGraw-Hill Book Company, Inc., New York, 1951) pp. 1010.

$$\delta_{rq} = \begin{cases} 1 & \text{if } r = q\ , \\ 0 & \text{if } r = q. \end{cases}$$

The summation is carried out over all possible values of q, the integration is taken over the cross-section of the guide in the Z plane. Since $\Delta\psi$ is continuous over the region, substituting Eq. (3) into Eq. (1) and some manipulation yields

$$\sum_q \left(k^2 B_{rq} - L_q^2\, \delta_{rq}\right) A_q = 0, \tag{4}$$

where

$$\Delta \phi_q + L_q^2\, \phi_q = 0, \tag{5}$$

$$B_{rq} = \int\!\!\int_s \phi_r\, h^2\, \phi_q\, dS\ ,$$

and L_q is a constant. For a two dimensional problem, the subscript q denotes the general indices

$$m,\ n = 0, 1, 2, 3, \cdots\cdots\cdots.$$

If p indicates a specific pair of m, n for TE or TM mode, Eq. (4) can be solved for k^2_p by Feenberg's iterative approximate method as follows:[2]

First order:

$$\left(k^2_p\right)^{(1)} = L_p^2 / B_{pp},$$

Second order:

$$\left(k_p^2\right)^{(2)} = L_p^2 \Big/ \left\{ B_{pp} + \sum_{q \neq p} B_{pq} B_{qp} \Big/ \left[\frac{L_q^2}{\left(k_p^2\right)^{(1)}} - B_{qq} \right] \right\}, \tag{6}$$

higher order approximations and the expressions for the expansion coefficients A_q can be found in reference 2. A suitable set of orthonormal functions which are solutions of Eq. (5) and satisfy the pertinent boundary conditions for TE modes is

$$\phi_q^{(I)} = \sqrt{\epsilon_m \epsilon_n / ab} \quad \cos\,(m\pi x/a)\ \cos\,(n\pi y/b), \tag{7a}$$

where $\epsilon_{m,n} = 1$, if $m,n = 0$, and $\epsilon_{m,n} = 2$ if $m,n \neq 0$. The corresponding set of functions for TM modes is

$$\phi_q^{(II)} = \sqrt{4/ab} \quad \sin\,(m\pi y/b). \tag{7b}$$

The constant L_q^2 is given by

$$L_q^2 = L^2_{mn} = (m\pi/a)^2 + (n\pi/b)^2 \tag{8}$$

The approximate eigenfunction ψ_p may be obtained from Eq. (3) in which the expansion coefficients A_q can be calculated by substituting the approximate value of k_p^2 as shown in reference 2.

In some cases, the successive approximation of Eq. (6) is slowly convergent or not convergent at all. Under those conditions, other methods have to be applied.

The Rayleigh-Ritz Method

The difficulties resulting from non-convergence can be avoided by the Rayleigh-Ritz method.[4,5] The application of this method to the present problem will be described next.

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4 H. Sagan, Boundary and Eigenvalue Problems in Mathematical Physics (John Wiley and Son, Inc., New York, 1961) Chapter 3

5 R. Collin, Field Theory of Guided Waves (McGraw-Hill Book Company, Inc., New York, 1960) Chapter 6

It can be shown that

$$k^2 = - \frac{\iint_s \psi \Delta \psi \, dS}{\iint_s h^2 \psi^2 \, dS} , \tag{9}$$

where the eigenvalue k is a stationary quantity. It is a minimum if the corresponding eigenfunction ψ is a solution of Eq. (1) and subject to the boundary conditions as stated in Eqs. (2).

If the eigenfunction ψ is approximated by $\bar{\psi}$, then the corresponding approximate eigenvalue

$$\lambda = - \frac{\iint_s \bar{\psi} \Delta \bar{\psi} \, dS}{\iint_s h^2 \bar{\psi}^2 \, dS} \tag{10}$$

where $\bar{\psi}$ satisfies the same boundary conditions as ψ does. The function $\bar{\psi}$ can be written as a finite series

$$\bar{\psi} = \sum_q^Q A_q \phi_q . \tag{11}$$

The function ϕ_q is given by Eqs. (7), where the subscript q denotes a pair of indices m, n. The single sum then actually represents a finite double sum

$$\sum_q^Q = \sum_m^M \sum_n^N$$

with M, N being integers. Under this condition, $\lambda > k^2$.

Since the quantity λ of Eq. (10) is stationary, the scalar function $\bar{\psi}$ of Eq. (10) has to be adjusted such that the quotion on the right hand side becomes a minimum. By definition [Eq. (11)], the function $\bar{\psi}$ has to be adjusted by varying the coefficient A_q's only. It follows

$$\frac{\partial \lambda}{\partial A_r} = 0, \quad \text{for r equal to all possible values of q} .$$

Substituting Eq. (11) into Eq. (10) and taking the partial differential with respect to A_r yields

$$\sum_{q}^{Q} (L_q^2 \delta_{rq} - \lambda B_{rq}) A_q = 0, \tag{12}$$

where the constant L_q is defined by Eq. (8). Note that Eq. (12) is similar to Eq. (4) except the summation is summing over a finite number of terms instead of an infinite. The consequence is that λ remains an approximation. Since r can be taking on any pair of indices in q, Eq. (12) is a system of T linear homogeneous equations, where T is the total number of terms in Eq. (11). In order to have a nontrivial solution for the A_q's, the determinent formed from the coefficients within the parentheses vanishes. Therefore,

$$\det. \left| L_q^2 \delta_{rq} - B_{rq} \lambda \right| = 0 \tag{13}$$

Eq. (13) is an algebric equation for λ of order T. Since the matrix form of the determinant is real and symmetric, it always can be solved for T real roots of λ by Newton's Method[6] or by an electronic computer. If $\lambda_i^{(T)}$ denotes the i^{th} root calculated by a T x T determinant of the form as Eq. (13) in which all the lower order elements are included, and all roots are distinct, i.e. $\lambda_1 < \lambda_2 < \lambda_3 \ldots\ldots\ldots < \lambda_T$, it can be shown that

$$\lambda_i^{(i)} > \lambda_i^{(i+1)} > \lambda_i^{(i+2)} > \ldots\ldots > \lambda_i^{(T)} > \ldots\ldots > k_i^2 \tag{14}$$

for all i, where k_i^2 is the i^{th} propagation constant of the waveguide. Since the function ψ can be expressed in terms of the complete infinite set of orthonormal functions $\{ \phi_q \}$ as shown in Eq. (3), it follows that

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[6] G. B. Thomas, Jr., Calculus and Analytic Geometry, (Addison-Wesley Publishing Company, Inc., 3rd ed. 1960) pp. 451

$$\lim_{T \to \infty} \lambda_i^{(T)} = k_i^2$$

The accuracy of the eigenvalue k_i^2 obtained by this method can be estimated by observing the convergence of $\lambda^{(T-1)}/\lambda_i^{(T)}$. The expansion coefficients A_q can be obtained by substituting λ_i into Eq. (12) and solving for A_q in terms of A_i.

Parabolic Guide With Axial Vane

As an example of the application of the theory discussed previously, the parabolic guide with axial vane is considered in this section. The cross-section of this guide in the R-plane may be transformed into a rectangle in the Z-plane (see Fig. 2) by means of the transformation function

$$R = Z^2/a.$$

The scale factor may be obtained by taking the magnitude of the first differentiation of R with respect to Z as follows:

$$h^2 = 4\,(x^2 + y^2)/a^2 .$$

It is possible to translate the y-axis in such a manner that it is collinear with the boundary of the rectangle in the Z-plane as shown in Fig. 1 (b). However, for simplicity, instead of doing this, it can be shown that for a scale factor which is symmetric with respect to the y-axis, the eigenfunction can be expanded as follows:

Odd TM modes:

$$\psi = \sum_{m,n} A_{m,n} \sqrt{2/ab}\, \sin (m\pi x/a) \sin (n\pi y/b),$$

Even TE modes:

$$\psi = \sum_{m,n} A_{m,n} \sqrt{\epsilon_m \, \epsilon_n/2ab}\, \cos (m\pi x/a) \cos (n\pi y/b),$$

where the boundaries of the rectangle are given by $y = 0$, $y = b$, $x = -a$, and $x = a$.

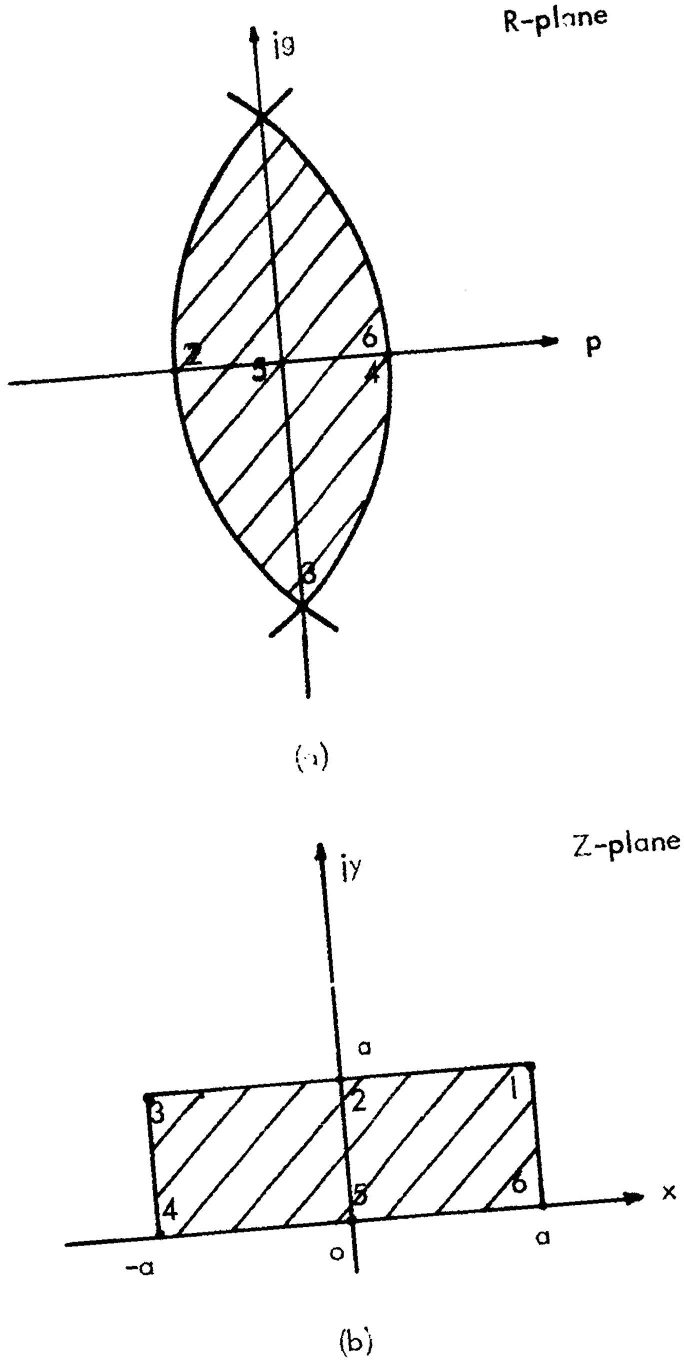

Fig. 2 - (a) Corresponding cross-sections Cross-section of a parabolic waveguide with axial vane in the R-plane. (b) Rectangular cross-section in the Z-plane.

Under this assumption, all integration is still taken over the cross-section of the rectangular guide, and

$$L^2_{mn} = (m\pi/a)^2 + (n\pi/b)^2.$$

For the discussed case, a = b. The even TM or the odd TE modes are not considered since no exact solutions are available for comparison.

Consider the oddTM mode first, the quantity B in both Eqs. (6) and (13) is given by

$$B_{pqpq} = (8/3) - [(2/p^2) + (2/q^2)]/\pi^2,$$

$$B_{qpsp} = B_{pqps} = (-1)^{p+r}\, 8\,\pi^{-2} [(q-s)^{-2} - (q+s)^{-2}]. \quad s \neq p$$

Several approximate eigenvalues of ka calculated by the Feenberg and Rayleigh-Ritz method for the odd $TM_{1,1}$ mode are tabulated in Table 1 and Table 11, where the subscript 1,1 indicates only one variation in both x and y direction.

Table 1 - Propagation constants computed by Feenberg's method ~~and the exact value~~

$TM_{1,1}$	$(ka)^{(1)}$	$(ka)^{(2)}$	$(ka)^{(3)}$	Exact[7]
(odd)	2.955	2.762	2.788	2.78

Table 11 - Propagation constants computed by Rayleigh-Ritz method and the exact value

$TM_{1,1}$	$\sqrt{\lambda^{(1)}}a$	$\sqrt{\lambda^{(2)}}a$	$\sqrt{\lambda^{(3)}}a$	$\sqrt{\lambda^{(25)}}a$	$\sqrt{\lambda^{(36)}}a$	Exact[7]
(odd)	2.955	2.843	2.795	2.780911	2.780895	2.78

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7 The exact values is given by $J_{1/4}(ka) = 0$ for odd TM_{mm} modes, $J_{3/4}(ka) = 0$ for even TE_{mm} modes. The roots were found in E. Jahnke and F. Emde, Tables of Functions (Dover Publication, 4th edition, 1945)

For the odd $TM_{1,1}$ mode, the Feenberg's method gives rapidly convergent answer comparable to the exact value. Using only the three lowest order terms in Eq. (12), the error of Rayleigh-Ritz method is approximately 0.5%. The values of $\sqrt{\lambda^{(25)}}\, a$ and $\sqrt{\lambda^{(36)}}\, a$ were calculated by a 7094 computor and show that $ka = 2.780$ is correct to four digits.

Considering the case of even TE modes, the quantity B is given by

$$B_{oooo} = 8/3,$$

$$B_{pooo} = B_{opoo} = B_{oopo} = B_{ooop} = (-1)^p \; 8 \; \sqrt{2}/(p\pi)^2, \qquad p \neq 0$$

$$B_{pqpq} = (8/3) + 2\pi^{-2} (p^{-2} + q^{-2}) \;, \qquad p,q \neq 0$$

$$B_{popo} = B_{opop} = (8/3) + 2/(p\pi)^2, \qquad p \neq 0$$

$$B_{pqrq} = B_{qpqr} = (-1)^{p+r} \; 8 \; [\,(p-r)^{-2} + (p+r)^{-2}\,]\, \pi^{-2} \qquad p \neq r$$

The first three order of approximations for the propagation constant $k\,a$ of $TE_{1,1}$ mode calculated by the Feenberg method are tabulated with the exact value in Table III. The slow convergence results from the fact that the maximum of the longitudinal magnetic field of the even TE modes of the air filled rectangular guide is located at the same point where the permittivity and permeability of the non-uniformly filled rectangular guide is zero. For the same reason the propagation constants of the higher order even modes, like $TE_{2,2}$, $TE_{3,3}$.........etc., computed by this method are not convergent at all. Using the 7094 computer to solve the 36 x 36 secular determinant of the Rayleigh-Ritz method, the approximate value of ka for even $TE_{1,1}$ mode is 3.4913. This is an excellent solution in comparison with the exact value.

Table 111 - Propagation constants computed by Feenberg's method and the exact value

$TE_{1,1}$	$(ka)^{(1)}$	$(ka)^{(2)}$	$(ka)^{(3)}$	Exact[7]
(even)	2.536	3.685	3.185	3.49

Semi-Elliptic Waveguide

As another example of the Feenberg's method, the semi-elliptic waveguide will be investigated as follows:

The conformal transformation is

$$R = a \cos Z$$

and transforms the cross-section of a semi-elliptic waveguide in the R-plane into a rectangle in the Z-plane as shown in Fig. 3. The boundaries of the rectangle are at $y = 0$, $y = b$, $x = 0$, and $x = \pi$. The scale factor h is given by

$$h^2 = a^2 (\cosh^2 y - \cos^2 x).$$

The quantity B is given by

TM:

$$B_{pqpq} = \frac{\text{Sinh } 2b}{4b} - b[\sinh 2b][4b^2 + (2q\pi)^2]^{-2} + (1/4)\delta_{p,1}$$

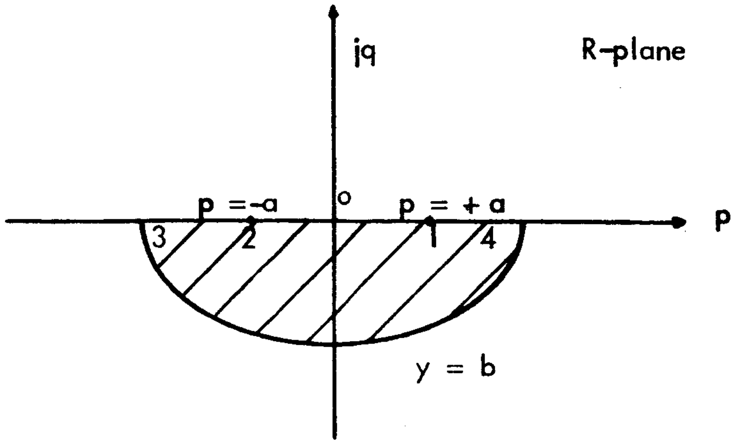
$$B_{pqpq} = (1/4)\ (\delta_{p+r,2} - \delta_{1p-41,2}),$$

$$B_{pqps} = (-1)^{q+s}\, b[\sinh 2b]\ \{[4b^2 + (q-s)^2\pi^2]^{-1} - [4b^2 + (q+s)^2\pi^2]^{-1}\}$$

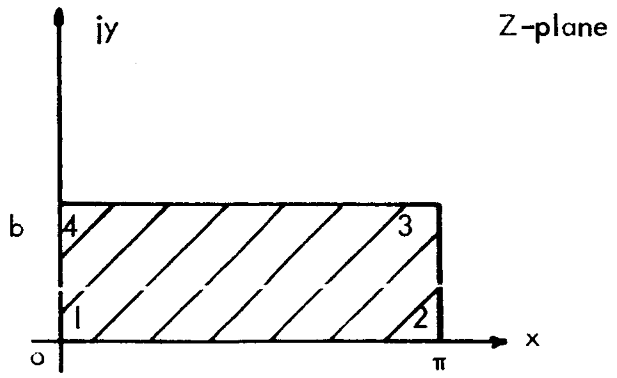
TE:

$$B_{oooo} = \sinh 2b/4b,$$

$$B_{popo} = (\sinh 2b/4b) - (1/4)\ \delta_{p,1}$$



(a)



(b)

Fig. 3 - Corresponding Cross-sections. (a) Semi-elliptic guide.
(b) Rectangular guide

$$B_{pqpq} = \{ (1/4b) + b \, [4b^2 + (2q\pi)^2]^{-1} \} \sinh 2b - \frac{\delta_{p,1}}{4}, \qquad q \neq 0$$

$$B_{pqrq} = -\frac{1}{4} (\delta_{p+r,2} + \delta_{|p-r|,2}),$$

$$B_{pqrq} = (-1)^{q+r} \, b \sinh 2b \, \{ [4b^2 + (q+s)^2\pi^2]^{-1} + [4b^2 + (q-s)^2\pi^2]^{-1} \}, \; q \neq s$$

$$B_{pqoq} = B_{oqpq} = -\frac{\sqrt{2}}{4} \, \delta_{p,2} \, ,$$

$$B_{pqpo} = B_{popq} = (-1)^q \sqrt{2} \;\; b \sinh 2b/(4b^2 + q^2\pi^2),$$

The propagation constants of $TM_{1,1}$, TM_{21} and TE_{10} modes are computed by the Feenberg method up to the 3rd order approximation. They are tabulated in Table IV, and compared with the exact values.

Table IV - Propagation constants computed by Feenberg's Method with exact values

	$(ka)^{(1)}$	$(ka)^{(2)}$	$(ka)^{(3)}$	Exact[8]
TM_{11}	3.182	3.073	3.082	3.0
TM_{21}	4.104	3.839	3.885	/
TE_{10}	1.234	1.203	1.204	1.2

Note that the convergence is very good for these three cases, but no convergence is obtained for the TE_{11} mode. This can be explained by the same reason as in the previous example.

[8] Calculated from curves given by L. J. Chu, J. Appl. Phys. vol 9, pp. 583, September 1938

Discussion

The cross-section of an arbitrarily-shaped waveguide is transformed into a rectangle. The equivalent rectangular guide is then filled with a nonuniform, anisotropic medium. The Feenberg perturbation method and the Rayleigh-Ritz method can be used for determining the propagation characteristics and the field components distribution. The latter method is preferable in cases where the convergence of the former method is not satisfactory. For higher-order modes, the Feenberg's method is simpler if the convergence is satisfactory. The computed examples show that good approximation can be achieved with small number of terms.

It should be mentioned that the arbitrary cross-section also can be transformed into a circle. The basic equations remain the same except the expansion of the eigenfunction ψ in terms of cylindrical functions. The expansion is

$$\psi = \sum_{m}\sum_{n} A_{mn} J_{m} (\alpha_{mn} r) \cos m\varphi ,$$

where TM: $J_m (\alpha_{mn} r_o) = 0$; TE: $J_m' (\alpha_{mn} r_o) = 0$, and r_o is the radius of the cylinderical conducting wall.

Acknowledgments

The authors wish to thank Miss Patricia Lucas and Mr. Rolf Duerr for their assistance in computing some of the data.